MUON DECAY

Consider the following muon decay process.

$$\mu^- \to \nu_\mu + \bar{\nu}_e + e^-$$

Define momentum and spinor vectors for each particle.

Particle	Symbol	Momentum	Spinor (up)	Spinor (down)
Muon	μ^-	p_1	u_{11}	u_{12}
Muon neutrino	ν_μ	p_2	u_{21}	u_{22}
Electron antineutrino	$\bar{\nu}_e$	p_3	v_{31}	v_{32}
Electron	e^-	p_4	u_{41}	u_{42}

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} \qquad p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} \qquad p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix} \qquad p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}$$

$$u_{11} = \begin{pmatrix} E_1 + m_1 \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix} \quad u_{21} = \begin{pmatrix} E_2 + m_2 \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix} \quad v_{31} = \begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E_3 + m_3 \\ 0 \end{pmatrix} \quad u_{41} = \begin{pmatrix} E_4 + m_4 \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}$$

$$u_{12} = \begin{pmatrix} 0 \\ E_1 + m_1 \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix} \quad u_{22} = \begin{pmatrix} 0 \\ E_2 + m_2 \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix} \quad v_{32} = \begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E_3 + m_3 \end{pmatrix} \quad u_{42} = \begin{pmatrix} 0 \\ E_4 + m_4 \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

The spin averaged probability density for the process is

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{G^2}{4N} \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \sum_{s_3=1}^{2} \sum_{s_4=1}^{2} \left| \left(\bar{u}_4 \gamma^\mu (1-\gamma^5) v_3 \right) \left(\bar{u}_2 \gamma_\mu (1-\gamma^5) u_1 \right) \right|^2 \\ &= \frac{G^2}{4} \operatorname{Tr} \left[\not{p}_4 \gamma^\mu (1-\gamma^5) \not{p}_3 \gamma^\nu (1-\gamma^5) \right] \operatorname{Tr} \left[\not{p}_2 \gamma_\mu (1-\gamma^5) \not{p}_1 \gamma_\nu (1-\gamma^5) \right] \\ &= 64 G^2 (p_1 \cdot p_3)(p_2 \cdot p_4) \end{aligned}$$

where s_j selects up or down and N is the following spinor normalization constant.

$$N = (E_1 + m_1)(E_2 + m_2)(E_3 + m_3)(E_4 + m_4)$$

Run "muon-decay-1.txt" to verify that

$$\begin{aligned} \frac{1}{N} \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \sum_{s_3=1}^{2} \sum_{s_4=1}^{2} \left| \left(\bar{u}_4 \gamma^\mu (1-\gamma^5) v_3 \right) \left(\bar{u}_2 \gamma_\mu (1-\gamma^5) u_1 \right) \right|^2 \\ = \operatorname{Tr} \left[\not{p}_4 \gamma^\mu (1-\gamma^5) \not{p}_3 \gamma^\nu (1-\gamma^5) \right] \operatorname{Tr} \left[\not{p}_2 \gamma_\mu (1-\gamma^5) \not{p}_1 \gamma_\nu (1-\gamma^5) \right] \end{aligned}$$

Run "muon-decay-2.txt" to verify that

$$\frac{1}{4} \operatorname{Tr} \left[\not{p}_4 \gamma^\mu (1-\gamma^5) \not{p}_3 \gamma^\nu (1-\gamma^5) \right] \operatorname{Tr} \left[\not{p}_2 \gamma_\mu (1-\gamma^5) \not{p}_1 \gamma_\nu (1-\gamma^5) \right] = 64 (p_1 \cdot p_3)(p_2 \cdot p_4)$$