Quantum electric field

Consider a light wave propagating in the z direction. For simplicity let the light be linearly polarized with electric field vector $\mathbf{E}$ pointing in the x direction.

$$\mathbf{E}(t,x,y,z) = \begin{pmatrix} E_x \cos(kz - \omega t) \\ 0 \\ 0 \end{pmatrix}$$

Symbol ω is angular frequency and k is the wave number $k = \omega / c$.

The corresponding wave function is

$$\psi = A \left|n - \tfrac{1}{2}\right\rangle + B \left|n + \tfrac{1}{2}\right\rangle$$

where n is the number of photons per unit volume and

$$A = \exp\left(-i\left(n - \tfrac{1}{2}\right)\omega t\right)$$
$$B = \exp\left(-i\left(n + \tfrac{1}{2}\right)\omega t\right)$$

The electric field operator is

$$\hat{\mathscr{E}} = C\hat{a} + C^*\hat{a}^\dagger$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the lowering and raising operators such that

$$a\left|n + \tfrac{1}{2}\right\rangle = \sqrt{n}\left|n - \tfrac{1}{2}\right\rangle$$
$$a^\dagger\left|n - \tfrac{1}{2}\right\rangle = \sqrt{n}\left|n + \tfrac{1}{2}\right\rangle$$

The quantity C is

$$C = \sqrt{\frac{\hbar\omega}{2V\varepsilon_0}}\exp(ikz)$$

where V is a unit volume.

Apply electric field operator $\hat{\mathscr{E}}$ to wave function ψ.

$$\begin{aligned} \hat{\mathscr{E}}\psi &= C\hat{a}\psi + C^*\hat{a}^\dagger\psi \\ &= CA\sqrt{n-1}\left|n - \tfrac{3}{2}\right\rangle + CB\sqrt{n}\left|n - \tfrac{1}{2}\right\rangle + C^*A\sqrt{n}\left|n + \tfrac{1}{2}\right\rangle + C^*B\sqrt{n+1}\left|n + \tfrac{3}{2}\right\rangle \end{aligned}$$

The observed electric field is the eigenvalue $\mathscr{E}$ such that $\hat{\mathscr{E}}\psi = \mathscr{E}\psi$.

$$\begin{aligned} \mathscr{E} &= \psi^\dagger\hat{\mathscr{E}}\psi \\ &= \left\langle n - \tfrac{1}{2}\right| A^*CB\sqrt{n}\left|n - \tfrac{1}{2}\right\rangle + \left\langle n + \tfrac{1}{2}\right| B^*C^*A\sqrt{n}\left|n + \tfrac{1}{2}\right\rangle \\ &= \sqrt{n}\,C\exp(-i\omega t) + \sqrt{n}\,C^*\exp(i\omega t) \\ &= \sqrt{\frac{2n\hbar\omega}{V\varepsilon_0}}\cos(kz - \omega t) \end{aligned}$$

Identifying $\mathscr{E}$ as the first component of $\mathbf{E}$ we have $\mathscr{E} = E_x\cos(kz - \omega t)$. Hence the electric field amplitude E_x is proportional to the square root of photon density.

$$E_x = \sqrt{\frac{2n\hbar\omega}{V\varepsilon_0}}$$

Run “quantum-electric-field-1.txt” to verify.

The SI unit of electric field strength is volts per meter. The script “quantum-electric-field-2.txt” calculates the SI constant for converting photon density to volts per meter. Yellow light with wavelength $\lambda = 600$ nanometers is used for angular frequency ω. The result is

$$\sqrt{\frac{2n\hbar\omega}{V\varepsilon_0}} = 2.7 \times 10^{-4}\,\text{volt meter}^{-1} \times \sqrt{n}$$

The symbol V is a one cubic meter unit volume. The script also converts volts per meter to base units.

$$1\,\text{volt meter}^{-1} = 1\,\text{kilogram meter ampere}^{-1}\,\text{second}^{-3}$$

Reference

Dommelen, Leon van. “Quantum Mechanics for Engineers, Section A.23 Quantization of radiation.” http://www.eng.fsu.edu/~dommelen/quantum/style_a/qftqem.html