Recall that the following wave functions are solutions to the Dirac equation.

$$\psi_1 = \begin{pmatrix} \omega+m \\ 0 \\ k_z \\ k_x+ik_y \end{pmatrix} \exp[i(k_x x + k_y y + k_z z - \omega t)] \quad \psi_7 = \begin{pmatrix} k_z \\ k_x+ik_y \\ \omega+m \\ 0 \end{pmatrix} \exp[-i(k_x x + k_y y + k_z z - \omega t)]$$

$$\psi_2 = \begin{pmatrix} 0 \\ \omega+m \\ k_x-ik_y \\ -k_z \end{pmatrix} \exp[i(k_x x + k_y y + k_z z - \omega t)] \quad \psi_8 = \begin{pmatrix} k_x-ik_y \\ -k_z \\ 0 \\ \omega+m \end{pmatrix} \exp[-i(k_x x + k_y y + k_z z - \omega t)]$$

A spinor is the vector part of a Dirac wave function. The following eight spinors are used for scattering calculations. The u spinors are fermions from ψ_1 and ψ_2. The v spinors are anti-fermions from ψ_7 and ψ_8. The last digit of the u or v subscript is 1 for spin up and 2 for spin down.

$$u_{11} = \begin{pmatrix} E_1+m_1 \\ 0 \\ p_{1z} \\ p_{1x}+ip_{1y} \end{pmatrix} \quad v_{21} = \begin{pmatrix} p_{2z} \\ p_{2x}+ip_{2y} \\ E_2+m_2 \\ 0 \end{pmatrix} \quad u_{31} = \begin{pmatrix} E_3+m_3 \\ 0 \\ p_{3z} \\ p_{3x}+ip_{3y} \end{pmatrix} \quad v_{41} = \begin{pmatrix} p_{4z} \\ p_{4x}+ip_{4y} \\ E_4+m_4 \\ 0 \end{pmatrix}$$

$$u_{12} = \begin{pmatrix} 0 \\ E_1+m_1 \\ p_{1x}-ip_{1y} \\ -p_{1z} \end{pmatrix} \quad v_{22} = \begin{pmatrix} p_{2x}-ip_{2y} \\ -p_{2z} \\ 0 \\ E_2+m_2 \end{pmatrix} \quad u_{32} = \begin{pmatrix} 0 \\ E_3+m_3 \\ p_{3x}-ip_{3y} \\ -p_{3z} \end{pmatrix} \quad v_{42} = \begin{pmatrix} p_{4x}-ip_{4y} \\ -p_{4z} \\ 0 \\ E_4+m_4 \end{pmatrix}$$

These are the associated momentum vectors.

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} \quad p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} \quad p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix} \quad p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}$$

Spinors are solutions to the following momentum-space Dirac equation with $\not{p} = p \cdot (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$.

$$(\not{p} - m)u = 0 \qquad (\not{p} + m)v = 0$$

Up and down spinors have the following "completeness property."

$$u_{11}\bar{u}_{11} + u_{12}\bar{u}_{12} = (E_1+m_1)(\not{p}_1 + m_1) \qquad v_{21}\bar{v}_{21} + v_{22}\bar{v}_{22} = (E_2+m_2)(\not{p}_2 - m_2)$$

The adjoint of a spinor is $\bar{u} = u^\dagger \gamma^0$. The adjoint is a row vector hence $u\bar{u}$ is an outer product.

This script verifies all eight spinors.

```
-- www.eigenmath.org/spinors.txt

-- Verify Dirac spinor properties.

E1 = sqrt(p1x^2 + p1y^2 + p1z^2 + m1^2)
E2 = sqrt(p2x^2 + p2y^2 + p2z^2 + m2^2)
E3 = sqrt(p3x^2 + p3y^2 + p3z^2 + m3^2)
E4 = sqrt(p4x^2 + p4y^2 + p4z^2 + m4^2)

p1 = (E1, p1x, p1y, p1z)
p2 = (E2, p2x, p2y, p2z)
p3 = (E3, p3x, p3y, p3z)
p4 = (E4, p4x, p4y, p4z)

u11 = (E1 + m1, 0, p1z, p1x + i p1y)
u12 = (0, E1 + m1, p1x - i p1y, -p1z)

v21 = (p2z, p2x + i p2y, E2 + m2, 0)
v22 = (p2x - i p2y, -p2z, 0, E2 + m2)

u31 = (E3 + m3, 0, p3z, p3x + i p3y)
u32 = (0, E3 + m3, p3x - i p3y, -p3z)

v41 = (p4z, p4x + i p4y, E4 + m4, 0)
v42 = (p4x - i p4y, -p4z, 0, E4 + m4)

I = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))

gmunu = ((1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1))

gamma0 = ((1,0,0,0),(0,1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma1 = ((0,0,0,1),(0,0,1,0),(0,-1,0,0),(-1,0,0,0))
gamma2 = ((0,0,0,-i),(0,0,i,0),(0,i,0,0),(-i,0,0,0))
gamma3 = ((0,0,1,0),(0,0,0,-1),(-1,0,0,0),(0,1,0,0))

gamma = (gamma0,gamma1,gamma2,gamma3)

pslash1 = dot(p1,gmunu,gamma)
pslash2 = dot(p2,gmunu,gamma)
pslash3 = dot(p3,gmunu,gamma)
pslash4 = dot(p4,gmunu,gamma)

"checking dirac equation (1=ok)"

dot(pslash1 - m1 I,u11) == 0
```

```
dot(pslash1 - m1 I,u12) == 0
dot(pslash2 + m2 I,v21) == 0
dot(pslash2 + m2 I,v22) == 0
dot(pslash3 - m3 I,u31) == 0
dot(pslash3 - m3 I,u32) == 0
dot(pslash4 + m4 I,v41) == 0
dot(pslash4 + m4 I,v42) == 0

"checking completeness (1=ok)"

adjoint(u) = dot(conj(u),gamma0)

outer(u11,adjoint(u11)) + outer(u12,adjoint(u12)) == (E1 + m1) (pslash1 + m1 I)
outer(v21,adjoint(v21)) + outer(v22,adjoint(v22)) == (E2 + m2) (pslash2 - m2 I)
outer(u31,adjoint(u31)) + outer(u32,adjoint(u32)) == (E3 + m3) (pslash3 + m3 I)
outer(v41,adjoint(v41)) + outer(v42,adjoint(v42)) == (E4 + m4) (pslash4 - m4 I)
```